THE HIERARCHY OF HIGHER ORDER SOLUTIONS OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. In this paper, we provide a simple method to generate higher order position solutions and rogue wave solutions for the derivative nonlinear Schrödinger equation. The formulae of these higher order solutions are given in terms of determinants. The dynamics and structures of solutions generated by this method are studied.

1. Introduction

The derivative nonlinear Schrödinger equation (DNLS)

$$iq_t - q_{xx} + i(|q|^2 q)_x = 0. \tag{1}$$

plays an important role in plasma physics and nonlinear optics. It not only dominates the evolution of small-amplitude Alfén waves in a low-β plasma [1–4], but also is used to describe the behavior of large-amplitude magnetohydrodynamic (MHD) waves in a high-β plasma [5,6]. On the other hand, the DNLS equation governs the transmission of sub-picosecond in single mode optical fibers [7–9].

For the DNLS equation with vanishing boundary condition (VBC), Kaup and Newell (KN) [10] first found the one-soliton solution by inverse scattering transformation (IST). On the basis of bilinear transformation, the first N-soliton formula was obtained by Nakamuro and Chen [11]. Determinant expression of the N-soliton solution can be established via applying the Darboux transformation [12]. In the case of the non-vanishing boundary condition(NVBC), Kawata and Inoue developed an IST for the DNLS equation and obtained a breather-type soliton (paired soliton) [13]. Wadati *et al* derived the stationary solution for the DNLS equation under the plane wave boundary [14]. Chen and Lam [15] revised the IST for the DNLS by introducing an affine parameter, and derived a breather solution, which can be degenerated to both dark soliton and bright soliton.

Recently, rogue wave, an emerging phenomenon, is passionately discussed. The concept of rogue wave was first proposed in the studies of deep ocean waves [16, 17], and gradually extended to other fields such as optics fibre [18–20], Bose-Einstein condensates [21], capillary phenomena [22], and so on. Rogue wave, "which appears from nowhere and disappears without a trace (WANDT)" [23], possesses the following two remarkable characteristics: i) locates in both space and time, ii) exhibits a dominant peak.

The first order rogue wave was found in 1983 by Peregrine [24], which is a solution of the NLS equation. It is usually called the Peregrine soliton, and has been observed experimentally in fiber [25], water tank [26] and multi-component plasma [27]. The first order rogue wave solution of the DNLS equation was first found by Xu and coworkers [28] by the Darboux transformation

and certain limit technique. Recently, Guo et al [29] obtained two kinds of generalized Darboux transformations, and got the formulae of higher order solutions for both the VBC and NVBC. Moveover, Guo showed two patterns (fundamental and triangular) of the second order rogue wave from a special seed solution $q = \exp(-\mathrm{i}x)$, which are similar to the case of the NLS equation. Therefore, it is nature to ask whether the rogue wave solution of the DNLS equation possesses new structures that have not been found in other soliton equations such as the NLS equation.

The Darboux transformation, generated from the work of Darboux in 1882 for the Sturm-Liouville equation, has been an important method in generating solutions of integrable systems. To get the rogue wave solution, we need to iterate Darboux transformation at the same eigenvalue, but it does not work in this case. So we must modify the Darboux transformation to get the solutions at the same eigenvalue. In this paper, we adopt the Taylor expansion to deal with this defect, and obtain positon solutions, rational traveling solutions and rogue wave solutions.

The organization of this paper is as follows. In section 2, we provide a new method to generate solutions at the same eigenvalue base on the method of Darboux transformation and Taylor expansion, and display the formula of N-th order solution in terms of determinant. As applications, several explicitly analytic expressions are given, which include positon solutions, rational traveling solutions and rogue wave solutions. In section 3, we obtain the multi-rogue wave solutions by altering the mixed coefficients of eigenfunctions, which contain several free parameters. With the help of those parameters, we consider the dynamics of multi-rogue wave. Moreover, three kinds of new structures: *modified-triangular* structure, *ring-triangle* structure and *multi-ring* structure are given. The conclusion is given in the last section.

2. **The solutions of the DNLS equation**

The Kaup and Newell (KN) system [10]:

$$
\begin{cases}
r_t - \mathrm{i}r_{xx} - (r^2 q)_x = 0, \\
q_t + \mathrm{i}q_{xx} - (r q^2)_x = 0.
\end{cases}
\tag{2}
$$

can be represented as the integrability condition of the following Kaup and Newell spectral system (Lax pair) [10, 30]:

$$
\begin{cases}
\Psi_x = M\Psi = (J\lambda^2 + Q\lambda)\Psi, \\
\Psi_t = N\Psi = (2J\lambda^4 + V_3\lambda^3 + V_2\lambda^2 + V_1\lambda)\Psi,
\end{cases}
\tag{3}
$$

with

$$
J = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad
Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},
$$

$$
V_3 = 2Q, \quad V_2 = Jqr, \quad
V_1 = \begin{pmatrix} 0 & -\mathrm{i}q_x + q^2 r \\ \mathrm{i}r_x + r^2 q & 0 \end{pmatrix},
$$

Where $\lambda \in \mathbb{C}$, $\Psi \in \mathbb{C}^2$, Ψ is called the eigenfunction of the spectral problem (3) corresponding to eigenvalue λ.

When

$$
r = -q^*,
\tag{4}
$$

the KN system can be reduced to the DNLS equation, asterisk denotes complex conjugation.

2.1 Determinant expression

The N-th Darboux transformation of the KN system in terms of determinant was obtained In [28]. And the formulae for N-th order solutions were given as following:

Lemma 1. [28] Let $\Psi_i = \begin{pmatrix} f_i \\ g_i \end{pmatrix}$ $(i = 1, 2, \cdots, n)$ be distinct solutions related to λ_i of the spectral problem (3), then $(q^{[n]}, r^{[n]})$ given by the following formulae are new solutions of the KN system (2).

$$q^{[n]} = \frac{\Omega_{11}^2}{\Omega_{21}^2} q + 2\mathrm{i} \frac{\Omega_{11}\Omega_{12}}{\Omega_{21}^2}, \quad r^{[n]} = \frac{\Omega_{21}^2}{\Omega_{11}^2} r - 2\mathrm{i} \frac{\Omega_{21}\Omega_{22}}{\Omega_{11}^2}. \tag{5}$$

with $n = 2k$,

$$\Omega_{11} = \begin{vmatrix} \lambda_1^{n-1}g_1 & \lambda_1^{n-2}f_1 & \lambda_1^{n-3}g_1 & \cdots & \lambda_1 g_1 & f_1 \\ \lambda_2^{n-1}g_2 & \lambda_2^{n-2}f_2 & \lambda_2^{n-3}g_2 & \cdots & \lambda_2 g_2 & f_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1}g_n & \lambda_n^{n-2}f_n & \lambda_n^{n-3}g_n & \cdots & \lambda_n g_n & f_n \end{vmatrix},$$

$$\Omega_{12} = \begin{vmatrix} \lambda_1^{n}f_1 & \lambda_1^{n-2}f_1 & \lambda_1^{n-3}g_1 & \cdots & \lambda_1 g_1 & f_1 \\ \lambda_2^{n}f_2 & \lambda_2^{n-2}f_2 & \lambda_2^{n-3}g_2 & \cdots & \lambda_2 g_2 & f_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n}f_n & \lambda_n^{n-2}f_n & \lambda_n^{n-3}g_n & \cdots & \lambda_n g_n & f_n \end{vmatrix},$$

$$\Omega_{21} = \begin{vmatrix} \lambda_1^{n-1}f_1 & \lambda_1^{n-2}g_1 & \lambda_1^{n-3}f_1 & \cdots & \lambda_1 f_1 & g_1 \\ \lambda_2^{n-1}f_2 & \lambda_2^{n-2}g_2 & \lambda_2^{n-3}f_2 & \cdots & \lambda_2 f_2 & g_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1}f_n & \lambda_n^{n-2}g_n & \lambda_n^{n-3}f_n & \cdots & \lambda_n f_n & g_n \end{vmatrix},$$

$$\Omega_{22} = \begin{vmatrix} \lambda_1^{n}g_1 & \lambda_1^{n-2}g_1 & \lambda_1^{n-3}f_1 & \cdots & \lambda_1 f_1 & g_1 \\ \lambda_2^{n}g_2 & \lambda_2^{n-2}g_2 & \lambda_2^{n-3}f_2 & \cdots & \lambda_2 f_2 & g_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n}g_n & \lambda_n^{n-2}g_n & \lambda_n^{n-3}f_n & \cdots & \lambda_n f_n & g_n \end{vmatrix}.$$

Here $\lambda_{2l} = -\lambda_{2l-1}^*$ and $\Psi_{2l} = \begin{pmatrix} f_{2l} \\ g_{2l} \end{pmatrix} = \begin{pmatrix} g_{2l-1}^* \\ f_{2l-1}^* \end{pmatrix}.$

It is trivial to check $r^{[2k]} = -q^{[2k]^*}$.

2.2 Solutions from vacuum

Let us consider the trivial case. When $q = r = 0$, the following Ψ is an eigenfunction for λ,

$$\Psi = \begin{pmatrix} f \\ g \end{pmatrix}, \quad f = \exp(\mathrm{i}(\lambda^2 x + 2\lambda^4 t)), \quad g = \exp(-\mathrm{i}(\lambda^2 x + 2\lambda^4 t)). \tag{6}$$

By applying the above formulae (5), we can get N-soliton solution of the DNLS equation from vacuum. To get new kinds of solutions, we set the eigenvalues share the same value, i.e., iterating the Darboux transformation at the same eigenvalue. However, the formulae (5) will be ineffective in this case. Next, we will use the Taylor expansion to generate the Darboux

transformation and get the formula of $q^{[n]}$ at the same eigenvalue as we have done for the case of the NLS equation [31].

At first, we define new functions $\Psi[i, j, k]$ for a general solution $\Psi = \Psi(\lambda)$ corresponding to λ as following:

$$\lambda^j \Psi = \Psi[i, j, 0] + \Psi[i, j, 1]\epsilon + \Psi[i, j, 2]\epsilon^2 + \cdots + \Psi[i, j, k]\epsilon^k + \cdots , \tag{7}$$

with

$$\Psi[i, j, k] = \frac{1}{k!} \frac{\partial^k (\lambda_i^j \Psi(\lambda_i))}{\partial \lambda_i^k}.$$

In particular

$$\Psi[1, 1, 0] = \lambda_1 \Psi(\lambda_1), \qquad \Psi[i, j, 0] = \lambda_i^j \Psi(\lambda_i).$$

Proposition 1. *Let $\lambda_1 = \alpha_1 + \mathrm{i}\beta_1$, $\lambda_2 = -\lambda_1^*$, then the following formula is the n-th($n = 2k$) solution of the DNLS equation generated at the same eigenvalue.*

$$q^{[n]} = 2\mathrm{i}\frac{\delta_{11}\delta_{12}}{\delta_{21}^2} \tag{8}$$

where

$$\delta_{11} = \begin{vmatrix} g[1, n-1, 0] & f[1, n-2, 0] & g[1, n-3, 0] & \cdots & g[1, 1, 0] & f[1, 0, 0] \\ g[2, n-1, 0] & f[2, n-2, 0] & g[2, n-3, 0] & \cdots & g[2, 1, 0] & f[2, 0, 0] \\ g[1, n-1, 1] & f[1, n-2, 1] & g[1, n-3, 1] & \cdots & g[1, 1, 1] & f[1, 0, 1] \\ g[2, n-1, 1] & f[2, n-2, 1] & g[2, n-3, 1] & \cdots & g[2, 1, 1] & f[2, 0, 1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g[1, n-1, k-1] & f[1, n-2, k-1] & g[1, n-3, k-1] & \cdots & g[1, 1, k-1] & f[1, 0, k-1] \\ g[2, n-1, k-1] & f[2, n-2, k-1] & g[2, n-3, k-1] & \cdots & g[2, 1, k-1] & f[2, 0, k-1] \end{vmatrix},$$

$$\delta_{12} = \begin{vmatrix} f[1, n, 0] & f[1, n-2, 0] & g[1, n-3, 0] & \cdots & g[1, 1, 0] & f[1, 0, 0] \\ f[2, n, 0] & f[2, n-2, 0] & g[2, n-3, 0] & \cdots & g[2, 1, 0] & f[2, 0, 0] \\ f[1, n, 1] & f[1, n-2, 1] & g[1, n-3, 1] & \cdots & g[1, 1, 1] & f[1, 0, 1] \\ f[2, n, 1] & f[2, n-2, 1] & g[2, n-3, 1] & \cdots & g[2, 1, 1] & f[2, 0, 1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f[1, n, k-1] & f[1, n-2, k-1] & g[1, n-3, k-1] & \cdots & g[1, 1, k-1] & f[1, 0, k-1] \\ f[2, n, k-1] & f[2, n-2, k-1] & g[2, n-3, k-1] & \cdots & g[2, 1, k-1] & f[2, 0, k-1] \end{vmatrix},$$

$$\delta_{21} = \begin{vmatrix} f[1, n-1, 0] & g[1, n-2, 0] & f[1, n-3, 0] & \cdots & f[1, 1, 0] & g[1, 0, 0] \\ f[2, n-1, 0] & g[2, n-2, 0] & f[2, n-3, 0] & \cdots & f[2, 1, 0] & g[2, 0, 0] \\ f[1, n-1, 1] & g[1, n-2, 1] & f[1, n-3, 1] & \cdots & f[1, 1, 1] & g[1, 0, 1] \\ f[2, n-1, 1] & g[2, n-2, 1] & f[2, n-3, 1] & \cdots & f[2, 1, 1] & g[2, 0, 1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f[1, n-1, k-1] & g[1, n-2, k-1] & f[1, n-3, k-1] & \cdots & f[1, 1, k-1] & g[1, 0, k-1] \\ f[2, n-1, k-1] & g[2, n-2, k-1] & f[2, n-3, k-1] & \cdots & f[2, 1, k-1] & g[2, 0, k-1] \end{vmatrix}.$$

Proof. For the entries in the first column of Ω_{11} (5),

$$\lambda_1^{n-1}g_1 = g[1, n-1, 0],$$
$$\lambda_2^{n-1}g_2 = g[2, n-1, 0],$$
$$\lambda_3^{n-1}g_3 = g[1, n-1, 0] + g[1, n-1, 1]\epsilon,$$
$$\lambda_4^{n-1}g_4 = g[2, n-1, 0] + g[2, n-1, 1]\epsilon,$$
$$\vdots$$
$$\lambda_{n-1}^{n-1}g_{n-1} = g[1, n-1, 0] + g[1, n-1, 1]\epsilon + \cdots + g[1, n-1, k-1]\epsilon^{k-1},$$
$$\lambda_n^{n-1}g_n = g[2, n-1, 0] + g[2, n-1, 1]\epsilon + \cdots + g[2, n-1, k-1]\epsilon^{k-1}.$$

Taking the similar procedure to the other entries in Ω_{11}, Ω_{12}, and Ω_{21}. Finally, the $q^{[n]}$ can be obtained through simple calculation. $\square$

For example, when $n = 4$,

$$q^{[4]} = \frac{\delta_{11}^2}{\delta_{21}^2}q + 2\mathrm{i}\frac{\delta_{11}\delta_{12}}{\delta_{21}^2} = 2\mathrm{i}\frac{\delta_{11}\delta_{12}}{\delta_{21}^2}, \tag{9}$$

where

$$\delta_{11} = \begin{vmatrix} g[1,3,0] & f[1,2,0] & g[1,1,0] & f[1,0,0] \\ g[2,3,0] & f[2,2,0] & g[2,1,0] & f[2,0,0] \\ g[1,3,1] & f[1,2,1] & g[1,1,1] & f[1,0,1] \\ g[2,3,1] & f[2,2,1] & g[2,1,1] & f[2,0,1] \end{vmatrix},$$

$$\delta_{12} = \begin{vmatrix} f[1,4,0] & f[1,2,0] & g[1,1,0] & f[1,0,0] \\ f[2,4,0] & f[2,2,0] & g[2,1,0] & f[2,0,0] \\ f[1,4,1] & f[1,2,1] & g[1,1,1] & f[1,0,1] \\ f[2,4,1] & f[2,2,1] & g[2,1,1] & f[2,0,1] \end{vmatrix},$$

$$\delta_{21} = \begin{vmatrix} f[1,3,0] & g[1,2,0] & f[1,1,0] & g[1,0,0] \\ f[2,3,0] & g[2,2,0] & f[2,1,0] & g[2,0,0] \\ f[1,3,1] & g[1,2,1] & f[1,1,1] & g[1,0,1] \\ f[2,3,1] & g[2,2,1] & f[2,1,1] & g[2,0,1] \end{vmatrix}.$$

Substituting the eigenfunction (6) into the formula (9), we obtain the positon solution

$$q^{[4]}_{positon} = \frac{L_1^* L_2}{L_1^2}, \tag{10}$$

where
$$L_1 = G_1 - iG_2,$$
$$L_2 = -16i\alpha_1\beta_1(\cos(F_2) + i\sin(F_2))\left(\left(\beta_1^3 + 4i\alpha_1^4\beta_1 x + 4i\alpha_1^2\beta_1^3 x - 32i\alpha_1^4\beta_1^3 t + 16i\alpha_1^6\beta_1 t - 48i\alpha_1^2\right.\right.$$
$$\left.\times\beta_1^5 t\right)\sinh(F_1) - (i\alpha_1^3 + 4\alpha_1^3\beta_1^2 x + 32\alpha_1^3\beta_1^4 t + 48\alpha_1^5\beta_1^2 t + 4\beta_1^4\alpha_1 x - 16\beta_1^6\alpha_1 t)\cosh(F_1)\bigg),$$
$$G_1 = \alpha_1^4 + \beta_1^4 + 256\,\alpha_1^8\beta_1^2 xt - 256\,\alpha_1^4\beta_1^6 xt + 256\,\alpha_1^6\beta_1^4 xt - 256\,\alpha_1^2\beta_1^8 xt + 512\,\alpha_1^2\beta_1^{10}t^2$$
$$+ 32\,\alpha_1^2\beta_1^6 x^2 + 32\,\alpha_1^6\beta_1^2 x^2 + 512\,\alpha_1^{10}\beta_1^2 t^2 + 2048\,\alpha_1^8\beta_1^4 t^2 + 3072\,\alpha_1^6\beta_1^6 t^2$$
$$+ 64\,\alpha_1^4\beta_1^4 x^2 + 2048\,\alpha_1^4\beta_1^8 t^2 + (\alpha_1^4 - \beta_1^4)\cosh(2F_1),$$
$$G_2 = -16\,\alpha_1^2\beta_1^4 x - 384\,\alpha_1^4\beta_1^4 t + 64\,\alpha_1^2\beta_1^6 t + 16\,\alpha_1^4\beta_1^2 x + (2\,\alpha_1^3\beta_1 + 2\,\alpha_1\beta_1^3)\sinh(2F_1)$$
$$+ 64\,\alpha_1^6\beta_1^2 t,$$
$$F_1 = 4\,\alpha_1\beta_1\left(4\,t\alpha_1^2 - 4\,t\beta_1^2 + x\right),$$
$$F_2 = 2\,\alpha_1^2 x + 4\,\alpha_1^4 t - 24\,t\alpha_1^2\beta_1^2 - 2\,\beta_1^2 x + 4\,\beta_1^4 t.$$

when $x \to \pm\infty$, $|q^{[4]}| = 0$, when $x = 0$, $t = 0$, $|q^{[4]}|^2 = 64\beta_1^2$. A simple analysis shows that it possesses phase shift compared with 2-rd soliton when $t \to \pm\infty$. After taking values as $\alpha_1 = 0.5$, $\beta_1 = 0.5$, the evolution of positon solution of the DNLS equation is shown in Fig. 1.

Next, if we set $\alpha_1 \to 0$ in above procedure, we will get the the second order rational traveling solution. With these parameters, we find that the general solution can be given in the same form as (10), but with the values for L_1 and L_2 written by

$$L_1 = G_1 - iG_2,$$
$$L_2 = -8i\beta_1\exp(2i\beta_1^2(2\beta_1^2 t - x))(-3i - 12\beta_1^2 x - 48\beta_1^4 t + 48i\beta_1^4 x^2 + 2304i\beta_1^8 t^2 - 768i\beta_1^6 xt$$
$$- 64\beta_1^6 x^3 + 4096\beta_1^{12}t^3 - 3072\beta_1^{10}xt^2 + 768\beta_1^8 tx^2),$$
$$G_1 = 3 + 4096\,\beta_1^{10}tx^3 - 24576\,\beta_1^{12}t^2 x^2 + 65536\,x\beta_1^{14}t^3 - 768\,\beta_1^6 xt + 96\,\beta_1^4 x^2 - 65536\,\beta_1^{16}t^4$$
$$+ 4608\,\beta_1^8 t^2 - 256\,\beta_1^8 x^4,$$
$$G_2 = 576\,\beta_1^4 t - 48\,\beta_1^2 x + 3072\,\beta_1^8 tx^2 - 256\,\beta_1^6 x^3 + 16384\,\beta_1^{12}t^3 - 12288\,\beta_1^{10}xt^2.$$

The dynamics of rational travelling solution of the DNLS equation with $\beta_1 = 0.3$ are shown in Fig. 2. Actually, it represents the interaction of two rational traveling solitons, and shares same properties with positon.

2.3. Solutions from periodic solution

Here, we will apply the method discussed above to generate solutions from periodic seed solution with the same eigenvalue. Moreover, we generate a hierarchy of rogue wave solutions.

We start with a general periodic solution

$$q = ce^{(i(ax+bt))}, \quad b = a(-c^2 + a), \quad a, c \in \mathbb{R}. \tag{11}$$

Substituting (11) into the spectral problem (3), we obtain the eigenfunction $\Psi = \begin{pmatrix} f \\ g \end{pmatrix}$ corresponding to the eigenvalue λ via applying the method of separation of variables and superposition principle.

$$
\begin{pmatrix} f(x,t,\lambda) \\ g(x,t,\lambda) \end{pmatrix} = \begin{pmatrix} D_1\omega_{11}^1(x,t,\lambda) + D_2\omega_{11}^2(x,t,\lambda) + D_1\omega_{12}^1{}^*(x,t,-\lambda^*) + D_2\omega_{12}^2{}^*(x,t,-\lambda^*) \\ D_1\omega_{12}^1(x,t,\lambda) + D_2\omega_{12}^2(x,t,\lambda) + D_1\omega_{12}^1{}^*(x,t,-\lambda^*) + D_2\omega_{12}^2{}^*(x,t,-\lambda^*) \end{pmatrix}.
$$
$$(12)$$

where

$$
\begin{cases} D_1 = 1, \\ D_2 = 1. \end{cases}
$$
$$(13)$$

$$
\begin{pmatrix} \omega_{11}^1(x,t,\lambda) \\ \omega_{12}^1(x,t,\lambda) \end{pmatrix} = \begin{pmatrix} \exp(c_1(x + 2\lambda^2 t + (-c^2+a)t) + \dfrac{1}{2}(\mathrm{i}(ax+bt))) \\ \dfrac{\mathrm{i}a - 2\mathrm{i}\lambda^2 + 2c_1}{2\lambda c}\exp(c_1(x + 2\lambda^2 t + bt) - \dfrac{1}{2}(\mathrm{i}(ax+bt))) \end{pmatrix},
$$

$$
\begin{pmatrix} \omega_{11}^2(x,t,\lambda) \\ \omega_{12}^2(x,t,\lambda) \end{pmatrix} = \begin{pmatrix} \exp(-c_1(x + 2\lambda^2 t + (-c^2+a)t) + \dfrac{1}{2}(\mathrm{i}(ax+bt))) \\ \dfrac{\mathrm{i}a - 2\mathrm{i}\lambda^2 - 2c_1}{2\lambda c}\exp(-c_1(x + 2\lambda^2 t + bt) - \dfrac{1}{2}(\mathrm{i}(ax+bt))) \end{pmatrix},
$$

$$
\omega^1(x,t,\lambda) = \begin{pmatrix} \omega_{11}^1(x,t,\lambda) \\ \omega_{12}^1(x,t,\lambda) \end{pmatrix}, \quad \omega^2(x,t,\lambda) = \begin{pmatrix} \omega_{11}^2(x,t,\lambda) \\ \omega_{12}^2(x,t,\lambda) \end{pmatrix},
$$

$$
c_1 = \frac{\sqrt{-a^2 - 4\lambda^4 - 4\lambda^2(c^2 - a)}}{2}.
$$

To get the rogue wave solutions, the value of n in formula (5) must be even. When $n = 2k$, we obtain the new expression of $q^{[n]}$.

Proposition 2. *Assuming* $\lambda_1 = \frac{1}{2}\sqrt{-c^2 + 2\,a} - \frac{1}{2}\mathrm{i}c$, $\lambda_2 = -\lambda_1^*$, *then* $q^{[n]}$ *given by following formula is the k-th rogue wave solution for the DNLS equation.*

$$
q^{[n]} = \frac{\delta_{11}^2}{\delta_{21}^2}q + 2\mathrm{i}\frac{\delta_{11}\delta_{12}}{\delta_{21}^2};
$$
$$(14)$$

where

$$
\delta_{11} = \begin{vmatrix} g[1,n-1,1] & f[1,n-2,1] & g[1,n-3,1] & \cdots & g[1,1,1] & f[1,0,1] \\ g[2,n-1,1] & f[2,n-2,1] & g[2,n-3,1] & \cdots & g[2,1,1] & f[2,0,1] \\ g[1,n-1,2] & f[1,n-2,2] & g[1,n-3,2] & \cdots & g[1,1,2] & f[1,0,2] \\ g[2,n-1,2] & f[2,n-2,2] & g[2,n-3,2] & \cdots & g[2,1,2] & f[2,0,2] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g[1,n-1,k] & f[1,n-2,k] & g[1,n-3,k] & \cdots & g[1,1,k] & f[1,0,k] \\ g[2,n-1,k] & f[2,n-2,k] & g[2,n-3,k] & \cdots & g[2,1,k] & f[2,0,k] \end{vmatrix},
$$

$$
\delta_{12} = \begin{vmatrix} f[1,n,1] & f[1,n-2,1] & g[1,n-3,1] & \cdots & g[1,1,1] & f[1,0,1] \\ f[2,n,1] & f[2,n-2,1] & g[2,n-3,1] & \cdots & g[2,1,1] & f[2,0,1] \\ f[1,n,2] & f[1,n-2,2] & g[1,n-3,2] & \cdots & g[1,1,2] & f[1,0,2] \\ f[2,n,2] & f[2,n-2,2] & g[2,n-3,2] & \cdots & g[2,1,2] & f[2,0,2] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f[1,n,k] & f[1,n-2,k] & g[1,n-3,k] & \cdots & g[1,1,k] & f[1,0,k] \\ f[2,n,k] & f[2,n-2,k] & g[2,n-3,k] & \cdots & g[2,1,k] & f[2,0,k] \end{vmatrix},
$$

$$\delta_{21} = \begin{vmatrix} f[1,n-1,1] & g[1,n-2,1] & f[1,n-3,1] & \cdots & f[1,1,1] & g[1,0,1] \\ f[2,n-1,1] & g[2,n-2,1] & f[2,n-3,1] & \cdots & f[2,1,1] & g[2,0,1] \\ f[1,n-1,2] & g[1,n-2,2] & f[1,n-3,2] & \cdots & f[1,1,2] & g[1,0,2] \\ f[2,n-1,2] & g[2,n-2,2] & f[2,n-3,2] & \cdots & f[2,1,2] & g[2,0,2] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f[1,n-1,k] & g[1,n-2,k] & f[1,n-3,k] & \cdots & f[1,1,k] & g[1,0,k] \\ f[2,n-1,k] & g[2,n-2,k] & f[2,n-3,k] & \cdots & f[2,1,k] & g[2,0,k] \end{vmatrix}.$$

Proof. For the entries in the first column of Ω_{11} (5),

$$\lambda_1^{n-1} g_1 = g[1,n-1,1]\epsilon,$$
$$\lambda_2^{n-1} g_2 = g[2,n-1,1]\epsilon,$$
$$\lambda_3^{n-1} g_3 = g[1,n-1,1]\epsilon + g[1,n-1,2]\epsilon^2,$$
$$\lambda_4^{n-1} g_4 = g[2,n-1,1]\epsilon + g[2,n-1,2]\epsilon^2,$$
$$\vdots$$
$$\lambda_{n-1}^{n-1} g_{n-1} = g[1,n-1,1]\epsilon + g[1,n-1,2]\epsilon^2 + \cdots + g[1,n-1,k]\epsilon^k,$$
$$\lambda_n^{n-1} g_n = g[2,n-1,1]\epsilon + g[2,n-1,2]\epsilon^2 + \cdots + g[2,n-1,k]\epsilon^k.$$

Taking the similar procedure to the other entries in Ω_{11}, Ω_{12}, and Ω_{21}. Finally, the $q^{[n]}$ can be obtained through simple calculation. $\square$

Here we point out that the formula (14) is different with the result of [29].
Next, we present some special examples, which have different structures.

- The first order rogue wave solution
  For $n = 2$, we get the first order rogue wave solution according the above formulae

$$q_{1_{rw}}^{[2]} = \frac{L_1^* L_2}{L_1^2} c \exp^{ia(x - tc^2 + ta)}, \tag{15}$$

  where

$$L_1 = e_1 + ie_2, \quad L_2 = e_3 + ie_4,$$
$$e_1 = -8\,t^2 c^2 a^3 + 12\,t^2 c^4 a^2 - 8\,xc^2 ta^2 + 8\,xc^4 ta - 2\,ax^2 c^2 - 6\,t^2 c^6 a - 1,$$
$$e_2 = 4\,atc^2 - 6\,tc^4 + 2\,xc^2,$$
$$e_3 = 8\,t^2 c^2 a^3 + 8\,xc^2 ta^2 - 12\,t^2 c^4 a^2 + 2\,ax^2 c^2 - 8\,xc^4 ta + 6\,t^2 c^6 a - 3,$$
$$e_4 = 12\,atc^2 - 6\,tc^4 + 2\,xc^2.$$

  A direct analysis shows when $x \to \infty$, $t \to \infty$, $q^{[2]} \to c^2$, the maximum module of $\left|q^{[2]}\right|^2$ is equal to $9c^2$ and locates at the origin. A 1-rogue wave with particular parameters is shown in Fig. 3
- High order rogue wave solutions
  Generally, the expression of rogue wave becomes more complicated with increasing n (an analytic expression of the second rogue wave solution is displayed in appendix A). Therefore, we use numerical simulations to discuss the high order rogue wave for convenient. We set $a = 1$ and $c = 1$ in the following.

When $n = 4$, we can obtain the second order rogue wave solution of the DNLS equation according to the formula (14).

$$q_{2_{rw}}^{[4]} = \frac{L_1^* L_2}{L_1^2} \exp(\mathrm{i}x), \tag{16}$$

with

$$
\begin{aligned}
L_1 =& 9 + 90\,x^2 - 12\,x^4 + 666\,t^2 + 180\,t^4 + 8\,x^6 + 8\,t^6 - 54\,\mathrm{i}x + 24\,\mathrm{i}tx^4 - 216\,x^2t^2 \\
& - 72\,xt + 24\,x^4t^2 + 48\,x^3t + 48\,xt^3 + 288\,\mathrm{i}t^2x + 24\,x^2t^4 - 24\,\mathrm{i}t^4x - 24\,\mathrm{i}x^5 \\
& - 48\,\mathrm{i}x^3 - 48\,\mathrm{i}t^2x^3 + 198\,\mathrm{i}t + 24\,\mathrm{i}t^5 + 48\,\mathrm{i}t^3x^2 + 336\,\mathrm{i}t^3, \\
L_2 =& 45 - 198\,x^2 - 60\,x^4 - 486\,t^2 - 60\,t^4 - 48\,\mathrm{i}x^3 + 528\,\mathrm{i}t^3 + 72\,\mathrm{i}t^5 - 414\,\mathrm{i}t + 8\,x^6 \\
& + 8\,t^6 + 72\,\mathrm{i}tx^4 + 144\,\mathrm{i}t^3x^2 + 24\,\mathrm{i}t^4x + 48\,\mathrm{i}t^2x^3 - 576\,\mathrm{i}t^2x - 288\,\mathrm{i}x^2t - 90\,\mathrm{i}x \\
& - 504\,x^2t^2 + 504\,xt + 24\,x^4t^2 - 144\,x^3t - 144\,xt^3 + 24\,\mathrm{i}x^5 + 24\,x^2t^4.
\end{aligned}
$$

Besides, we succeed in reaching the 7-order rogue wave by applying the above formula (14). Nevertheless, the analytical expression is too tedious, we omit it here. Their dynamical evolutions are shown in Fig. 4. From the figures, we find that the maximum height of the k-th order rogue wave is $(2k + 1)^2$. As remarked in [31, 32], there are $\frac{k(k+1)}{2} - 1$ local maxima on each side of the $x = 0$ line for k-order rogue wave of the NLS equation. However, there are only k small peak on the each side of the $t = 0$ line in Fig. 4 for k-order rogue wave solution of the DNLS equation. We may make a conjecture here that the central peak of rogue wave of the DNLS equation contain more energy than the NLS.

3. The dynamics of rogue wave with parameters

In above section, we assume that $D_1 = 1$ and $D_2 = 1$ (13). Actually, both D_1 and D_2 can be assumed as some new constants on the premise that (12) is the eigenfunction of spectral system (3). In this section, we set D_1 and D_2 as following:

$$
\begin{cases}
D_1 = \exp\left(-\mathrm{i}c_1(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})\right), \\
D_2 = \exp\left(\mathrm{i}c_1(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})\right).
\end{cases} \tag{17}
$$

Here $S_0, S_1, S_2, S_3, \cdots, S_{k-1} \in \mathbb{C}$. Although the terms with nonzero orders of ϵ in eq.(17) vanish in the $\epsilon \to 0$ limit, their coefficients $S_i(i = 0, 1, 2, \cdots, k - 1)$ have a crucial effect on the structure of higher order rogue wave. Depending on these parameters, we can obtain a variety of solutions of the same order. Finding these relative positions in terms of S_i $(i = 0, 1, 2, \cdots, k-1)$ is the subject of our analysis below.

3.1. Solutions with one parameter

In this subsection, we will discuss the dynamics of high-order rogue wave in detail. In the case that only one of the parameters is nonzero, four typical models are obtained: *fundamental pattern*, *triangular* structure, *ring* structure, and *modified-triangular* structure. Moreover, the *modified-triangular* structure has never been found in other equations.

- Fundamental pattern

  For $n = 2$, the first order rogue wave possesses only one parameter S_0, which is shown in Fig. 5. We observe that it is a translation of the solution in Fig. 3 relative

9

to the origin. Actually, it can shift the 1-order rogue wave solution to an arbitrary position on the (x,t)-plane, but it is trivial. So we omit this case for higher order solutions.

- Triangular structure

In this subsection, we set $S_i = 0$ except S_1. The resulting wave functions for orders $k = 2, 3, 4, 5, 6, 7$ are shown in Fig. 6. Remarkably, all higher order solutions display *triangular* structure. We observe triangle with three peaks in Fig. 6(a)(It has been obtained in Ref [29].), ten peaks in Fig. 6(b), fifteen peaks in Fig. 6(c), etc. All peaks within the triangle are first order rogue waves. So we can conclude that the *triangular* structure of an order k rogue wave solution is composed of $\frac{k(k+1)}{2}$ first order rogue waves, and it can be observed that successive k rows possessing k, $k-1$, $k-2$, $\cdots$, 1 peaks respectively. Evidently, the structure of the second order rogue wave is same as the result of the NLS equation, which is called triplet [33].

Another remarkable feature of these solutions is that, for the triangle of k-order($k > 2$) rogue wave solutions, the outer triangle is composed of $3k - 3$ first order rogue waves, and the inner triangle contains $\frac{k^2-5k+6}{2}$ first order rogue waves which is similar to the *triangular* structure of $(k-3)$-th order rogue wave solution. For example, the 7-th order rogue wave solution in Fig. 6(f) is composed of 28 first order rogue waves, 18 first order rogue waves locating on the outer shell, and the inner is similar to the triangle of the forth order rogue wave containing 10 first order rogue waves.

- Modified-triangular structure

Actually, the inner triangle structure can form a higher order rogue wave inversely by changing the appearance of (17). For instance, when $k = 5$, if we set

$$
\begin{pmatrix} f(x,t,\lambda) \\ g(x,t,\lambda) \end{pmatrix} = \begin{pmatrix} D_1\omega_{11}^1(x,t,\lambda) + D_1\omega_{11}^2(x,t,\lambda) + D_2\omega_{12}^{1\,*}(x,t,-\lambda^*) + D_2\omega_{12}^{2\,*}(x,t,-\lambda^*) \\ D_1\omega_{12}^1(x,t,\lambda) + D_1\omega_{12}^2(x,t,\lambda) + D_2\omega_{12}^{1\,*}(x,t,-\lambda^*) + D_2\omega_{12}^{2\,*}(x,t,-\lambda^*) \end{pmatrix},
$$

with

$$
\begin{cases} D_1 = \exp(-ic_1^2(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})), \\ D_2 = \exp(ic_1^2(S_0 + S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \cdots + S_{k-1}\epsilon^{k-1})). \end{cases}
$$

we will get a triangular structure with an second order rogue wave located in the center. It is remarkable that this structure has never been given before in nonlinear science, which is called *modified-triangular* structure. One with special parameter is shown in Fig. 7.

- Ring structure

If we assume $S_i = 0$ except S_{k-1}, we can get *ring* structures, which are shown in Fig. 8. They possess 1-order rogue waves and higher order rogue waves. Peaks locating on the outer shell of the ring are all first order rogue waves, and locating in the center of the ring are higher order rogue waves. Besides, the number of the first order rogue wave and the order of the inner higher order rogue wave increase according to the order of rogue wave. From those figures, we can conclude that there are $2k - 1$ first order rogue waves locating on the outer shell of *ring* structure of the k-th order rogue wave solution, and a WANDT of order $k - 2$ locates in the center of ring. Notably, this structure has never been displayed for the DNLS equation.

3.2. Solutions with more than one parameter

Generally, there are $k - 1$ free parameters for k-order rogue wave solution. As we discussed in previous subsection, the four basic models are depending on particular parameters. If there are two or more parameters which are non-zero, new models will be obtained. Moreover, higher order WANDT locating in the center of ring structure can be split into lower order waves.

- Ring-triangle
    When $k = 4$ with parameter $S_3 \neq 0$, we have gotten a *ring* structure with 7 1-order rogue wave solution located on the outer shell, and a 2-order rogue wave locating in the center in above section. Further more, if the parameter S_1 is also non-zero, the central higher order peak is split into a *triangular* structure and the outer shell remains the same. When $k = 5, 6, 7$, the similar structure is also displayed. These phenomenons are displayed in Fig. 9. Therefore, we are able to conclude that the central higher order rogue wave will be split into a *triangular* structure if $S_{k-1} \gg 0$ and $S_1 \neq 0$ for k-order WANDT.

- Multi-ring
    Similarly, the central higher order WANDT in *ring* structure can also be split into *ring* structure. For instance, when $k = 5$, we have observe a *ring* structure in Fig. 8(b). In this case, if we set $S_2 \neq 0$, the inner 3-order rogue wave can be split into a *ring* model. Its dynamics are shown in Fig. 10. When $k = 6$, if we assume $S_5 = 1 \times 10^8$ and $S_4 = 1 \times 10^6$, the inner higher order rogue wave is split into a *ring* structure with a second order rogue wave locating in the center. Its evolution is shown in Fig. 11. Indeed, we can continuing decomposing the inner structure with the help of another parameters. A new *multi-ring* model of the six order is displayed in Fig.12. From these figures, we find that both the outer shell and the middle shell are circular, and the inner shell is triangular (or circular). Naturally, the 7-order rogue wave solution possesses the same character except for the difference that the central peak can be split into both a triangle model and a ring pattern. They are shown in Fig. 13 and Fig. 14.

4. Conclusions

In this paper, we generate the formulae of higher order positon solution in proposition 1 and higher order rogue wave solution in proposition 2 for the DNLS equation at the same eigenvalue with the method of Taylor expansion and limit technique. By applying these formulae, we get positon solutions, rational traveling solutions and rogue wave solutions. These formulae are given in terms of determinants explicitly. Remarkably, the formula for rogue wave solutions is really effective in achieving the analytic expression and computer simulation of k-order rogue wave. Further more, we give rise to solutions with different structures are obtained by adjusting the free parameters D_1 and D_2 in our formula. With the help of these parameters, we study the dynamics of higher order rogue wave solutions. Overall, there are four basic models for higher order rogue wave. By choosing proper parameters, combination structures can be obtained. For example, *fundamental pattern, triangular* structure, *ring* structure,*modified-triangular* structure, *ring-triangle* structure, and *multi-ring* structure. The last three models have never been given before.

In the last part of this paper, we make a classification of higher order rogue wave of the DNLS equation. We found out that some basis structures (*fundamental pattern*, *triangular* structure, *ring* structure) also appear in other equations such as NLS. But the *modified-triangular* structure is unique to the DNLS equation.

Our results give an essential understanding of the relation of shift parameters with relative positions, which will be useful in other integrable equations such as Hirota equation, Gerdjikov-Ivanov equation, the Davey-Stewartson equation, and so on.

APPENDIX A. THE EXPRESSION OF THE SECOND ORDER ROGUE WAVE SOLUTION

Here we present the expression of the second order rogue wave solution

$$q_{2rw} = -\frac{L_1^* L_2}{L_1^2}\exp(\mathrm{i}\left(ax + \left(-c^2 + a\right)at\right)), \tag{18}$$

where
$$L_1 = e_1 + \mathrm{i}e_2, \qquad L_2 = e_3 + \mathrm{i}e_4,$$

$$\begin{aligned}
e_1 =& -9 + 72\,ac^4xt - 216\,a^2c^2xt - 108\,a^2c^4t^2 - 18\,ac^6t^2 + 216\,c^6xt - 54\,ac^2x^2 - 216\,a^3c^2t^2 \\
& + 1088\,a^3c^{12}x^3t^3 - 1536\,a^8c^6xt^5 + 6528\,a^6c^8x^2t^4 - 3264\,a^4c^{10}x^3t^3 - 1920\,a^7c^6x^2t^4 \\
& + 3456\,a^5c^8x^3t^3 - 456\,a^3c^{10}x^4t^2 + 648\,ac^{14}t^4 - 1280\,a^6c^6x^3t^3 + 912\,a^4c^8x^4t^2 - 2268\,a^2c^{12}t^4 \\
& - 480\,a^5c^6x^4t^2 + 4272\,a^3c^{10}t^4 + 96\,a^3c^8x^5t - 1296\,ac^{12}xt^3 - 4368\,a^4c^8t^4 - 96\,a^4c^6x^5t \\
& + 3456\,a^2c^{10}xt^3 + 1728\,a^5c^6t^4 - 4320\,a^3c^8xt^3 - 8\,a^3c^6x^6 + 864\,ac^{10}x^2t^2 - 216\,a^3c^{18}t^6 \\
& + 1296\,a^4c^{16}t^6 - 3456\,a^5c^{14}t^6 + 864\,a^3c^{16}xt^5 + 5184\,a^6c^{12}t^6 - 4320\,a^4c^{14}xt^5 - 4608\,a^7c^{10}t^6 \\
& + 9216\,a^5c^{12}xt^5 - 1368\,a^3c^{14}x^2t^4 + 2304\,a^8c^8t^6 - 10368\,a^6c^{10}xt^5 + 5472\,a^4c^{12}x^2t^4 - 512\,a^9c^6t^6 \\
& + 6144\,a^7c^8xt^5 - 8736\,a^5c^{10}x^2t^4 - 96\,a^3c^4x^3t + 24\,ac^6x^4 - 12\,a^2c^4x^4 - 36\,c^4x^2 - 324\,c^8t^2 \\
& - 192\,a^6c^4t^4 + 2496\,a^4c^6xt^3 - 1656\,a^2c^8x^2t^2 - 384\,a^5c^4xt^3 + 1296\,a^3c^6x^2t^2 - 240\,ac^8x^3t \\
& - 288\,a^4c^4x^2t^2 + 288\,a^2c^6x^3t,
\end{aligned}$$

$$\begin{aligned}
e_2 =& 6\,c^2(9\,x + 12\,c^4x^2ta - 24\,c^4xt^2a^2 + 48\,xt^2c^2a^3 + 24\,x^2ta^2c^2 - 180\,c^6t^2xa + 1776\,c^8a^4t^4x \\
& - 360\,c^{10}a^2t^3x^2 - 51\,tc^2 + 18\,ta + 4\,c^4a^2x^5 + 108\,c^8t^2x - 1008\,c^{10}a^4t^5 - 240\,c^6t^3a^2 - \\
& 108\,c^{14}a^2t^5 - 576\,c^6a^6t^5 - 64\,c^4t^3a^3 + 128\,c^4a^7t^5 + 1056\,c^8a^5t^5 + 32\,t^3c^2a^4 + 324\,c^8t^3a \\
& - 36\,c^6tx^2 + 504\,c^{12}a^3t^5 + 40\,c^4a^3tx^4 + 324\,c^{12}a^2t^4x - 1200\,c^{10}a^3t^4x - 336\,c^6a^3t^2x^3 \\
& + 160\,c^4a^4t^2x^3 - 44\,c^6a^2tx^4 + 4\,c^4x^3 - 108\,c^{10}t^3 - 960\,c^6a^4t^3x^2 + 184\,c^8a^2t^2x^3 \\
& + 320\,c^4a^5t^3x^2 + 4\,x^3c^2a - 1216\,c^6a^5t^4x + 320\,c^4a^6t^4x + 992\,c^8a^3t^3x^2),
\end{aligned}$$

$$e_3 = (e_5 - e_1)c,$$

$$\begin{aligned}
e_4 =& 6\,c^3(-15\,x + 108\,c^4x^2ta + 456\,c^4xt^2a^2 - 336\,xt^2c^2a^3 - 120\,x^2ta^2c^2 - 324\,c^6t^2xa + 2928\,c^8a^4t^4x \\
& - 360\,c^{10}a^2t^3x^2 + 21\,tc^2 - 90\,ta + 4\,c^4a^2x^5 + 108\,c^8t^2x - 1584\,c^{10}a^4t^5 - 384\,c^6t^3a^2 - 108\,c^{14}a^2t^5 \\
& - 1344\,c^6a^6t^5 + 544\,c^4t^3a^3 + 384\,c^4a^7t^5 + 2016\,c^8a^5t^5 - 288\,t^3c^2a^4 + 324\,c^8t^3a - 36\,c^6tx^2 \\
& + 648\,c^{12}a^3t^5 + 56\,c^4a^3tx^4 + 324\,c^{12}a^2t^4x - 1584\,c^{10}a^3t^4x - 464\,c^6a^3t^2x^3 + 288\,c^4a^4t^2x^3 \\
& - 44\,c^6a^2tx^4 + 4\,c^4x^3 - 108\,c^{10}t^3 - 1664\,c^6a^4t^3x^2 + 184\,c^8a^2t^2x^3 + 704\,c^4a^5t^3x^2 - 12\,x^3c^2a \\
& - 2496\,c^6a^5t^4x + 832\,c^4a^6t^4x + 1344\,c^8a^3t^3x^2),
\end{aligned}$$

$$\begin{aligned}
e_5 =& 36 + 288\,ac^4xt - 576\,a^2c^2xt + 288\,a^2c^4t^2 + 432\,ac^6t^2 + 864\,c^6xt - 144\,ac^2x^2 - 576\,a^3c^2t^2 \\
& + 1296\,a^2c^{12}t^4 - 4032\,a^3c^{10}t^4 + 4032\,a^4c^8t^4 - 1728\,a^2c^{10}xt^3 - 768\,a^5c^6t^4 + 3072\,a^3c^8xt^3 \\
& - 384\,a^3c^4x^3t - 48\,a^2c^4x^4 - 144\,c^4x^2 - 1296\,c^8t^2 - 768\,a^6c^4t^4 + 288\,a^2c^8x^2t^2 - 1536\,a^5c^4xt^3 \\
& + 576\,a^3c^6x^2t^2 - 1152\,a^4c^4x^2t^2 + 192\,a^2c^6x^3t.
\end{aligned}$$

With the help the formula (14), we can also obtain the expression of k-th (k=3,4,5,6,7) order rogue wave. Since they are too complicated to write down, we omit them.

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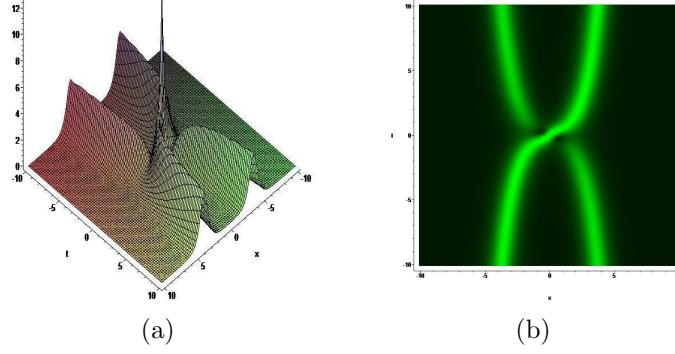[33] A Ankiewizc, D J Kedziora and N Akhmediev 2011 Phys. Lett. A **375**, 2782–2785.

(a)   (b)

FIGURE 1. (Color online) The dynamics of positon solution on the (x, t) plane
with $\alpha_1 = 0.5$, $\beta_1 = 0.5$.

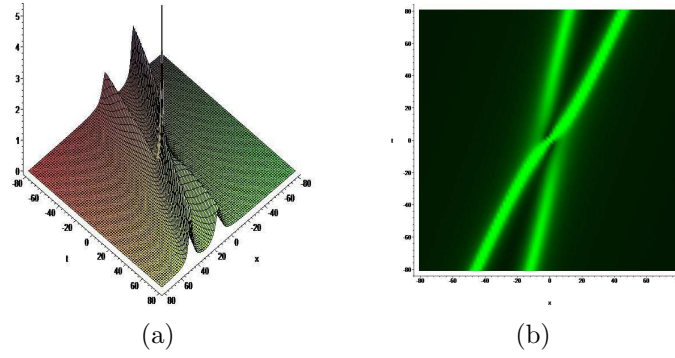(a)   (b)

FIGURE 2. (Color online) The dynamics of 2-rd rational traveling solution on
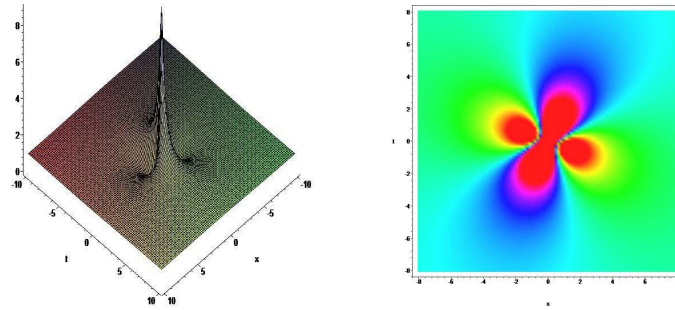the (x, t) plane with $\beta_1 = 0.3$.

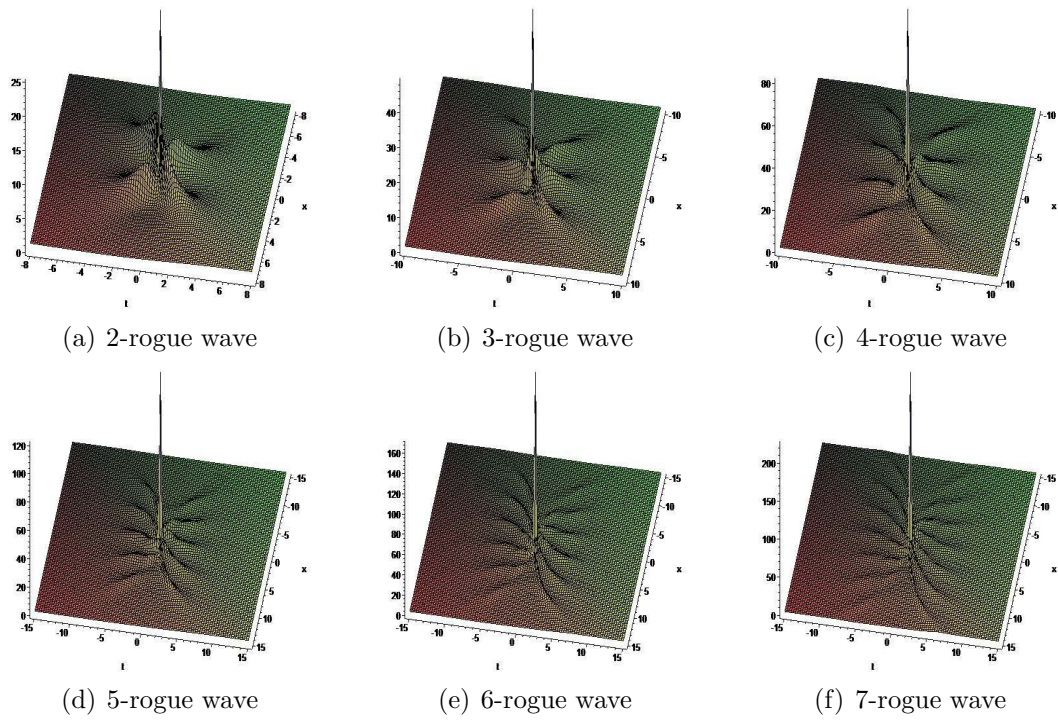FIGURE 3. 1-rogue wave solution with $a = 1$, $c = 1$.

(a) 2-rogue wave   (b) 3-rogue wave   (c) 4-rogue wave

(d) 5-rogue wave   (e) 6-rogue wave   (f) 7-rogue wave

FIGURE 4. (Color online)The dynamics of higher order rogue wave solution.

(a)　　　　　　　　　(b)

(c)　　　　　　　　　(d)
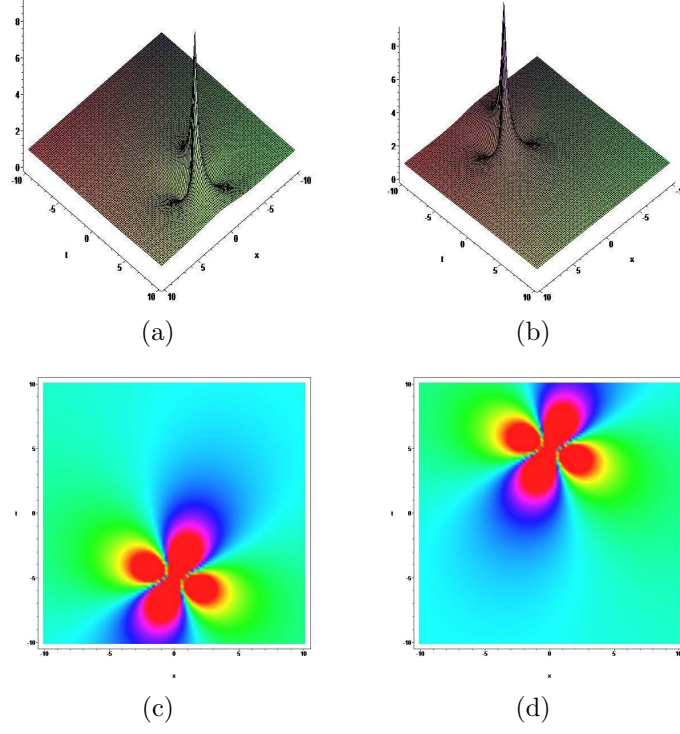
FIGURE 5. (Color online)Various form of 1-order rogue wave solution with particular parameter S_0. (a) The first order rogue wave with $S_0 = 5$, the maximum amplitude occurs at $x = 0$ and $t = -5$. (b) The first order rogue wave with $S_0 = -5$, the maximum amplitude occurs at $x = 0$ and $t = 5$

(a) 2-order rogue wave with $S_1 = 500$

(b) 3-order rogue wave with $S_1 = 500$

(c) 4-order rogue wave with $S_1 = 500$

(d) 5-order rogue wave with $S_1 = 500$

(e) 6-order rogue wave with $S_1 = 500$

(f) 7-order rogue wave with $S_1 = 250$
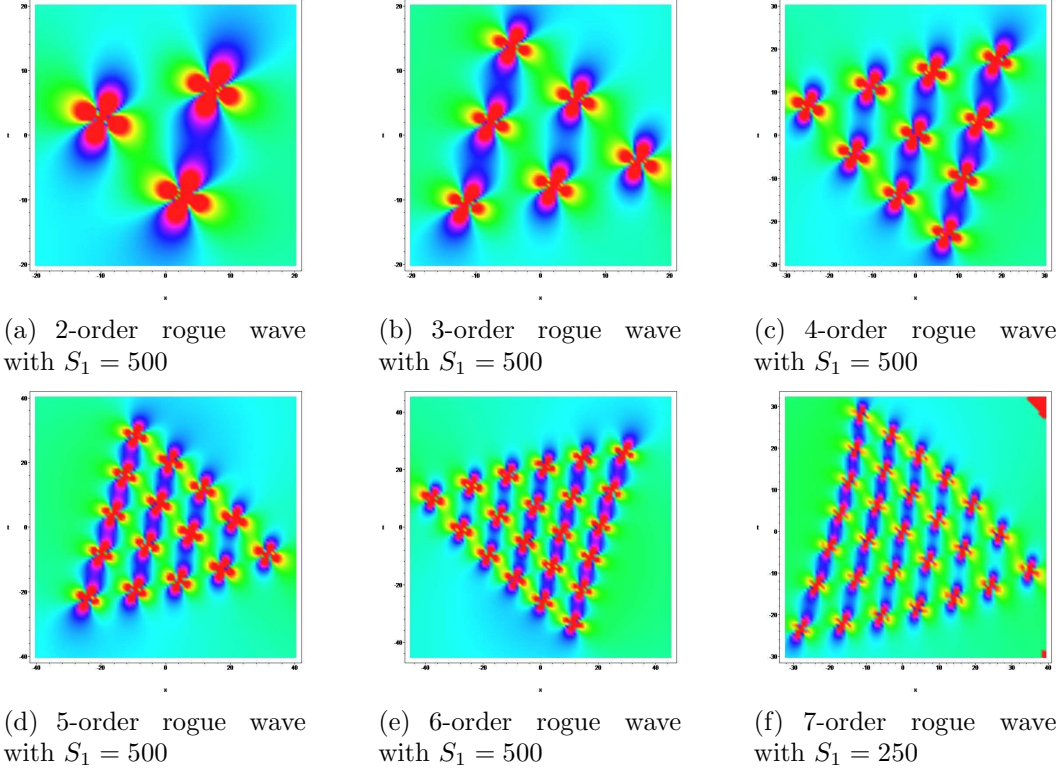
FIGURE 6. The *triangular* structures of higher order rogue waves.

(a)

(b)
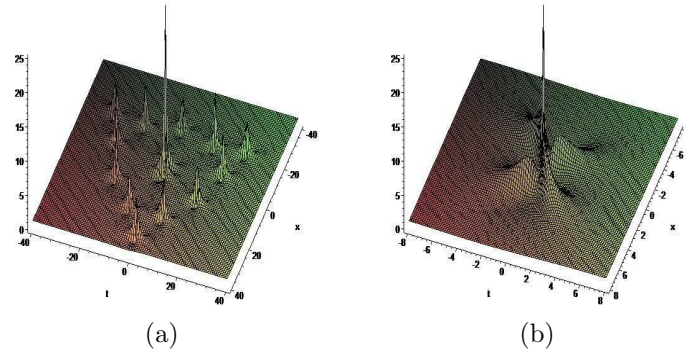
FIGURE 7. (Color online)The *modified-triangular* structure of 5-order rogue wave with a second order rogue wave located in the center. (a) An overall profile of 5-order rogue wave with $S_1 = 100$. (b) The centra profile of the right panel.
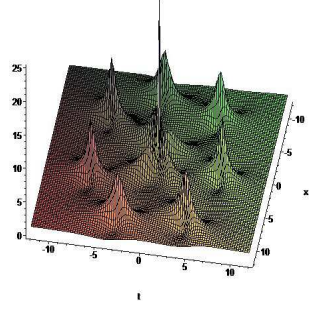
(a) 4-order rogue wave with $S_3 = 10000$
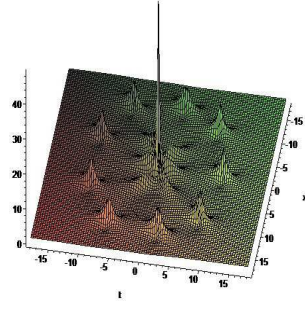
(b) 5-order rogue wave with $S_4 = 500000$
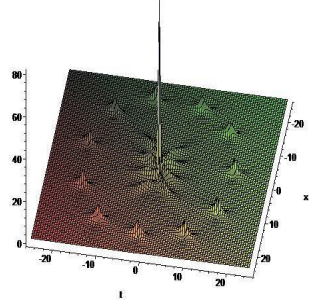
(c) 6-order rogue wave with $S_5 = 1 \times 10^8$
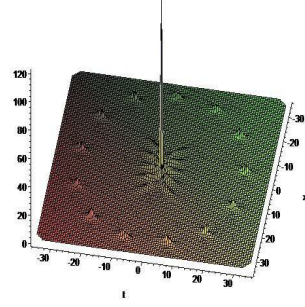
(d) 7-order rogue wave with $S_6 = 1 \times 10^{10}$

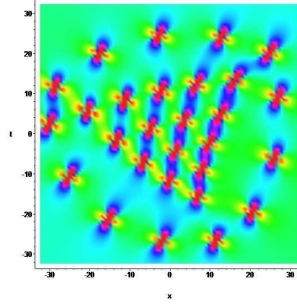FIGURE 8. (Color online) The *ring* structures of higher order rogue wave solutions with $S_{k-1} \neq 0$.
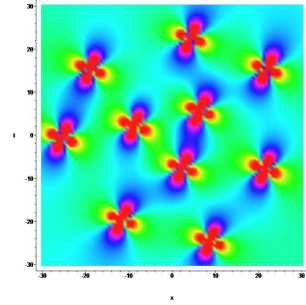
(a) 4-order rogue wave with $S_1 = 300$ and $S_3 = 1 \times 10^7$
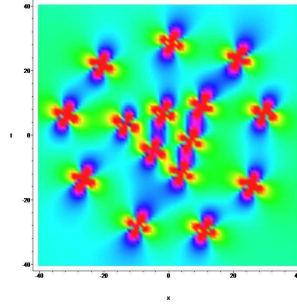
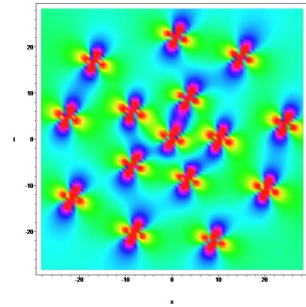(b) 5-order rogue wave with $S_1 = 200$ and $S_4 = 1 \times 10^9$
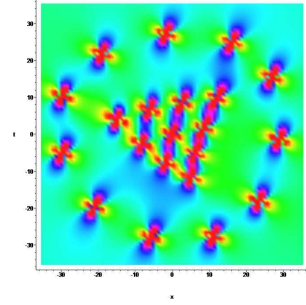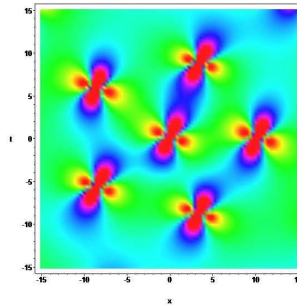
(c) 6-order rogue wave with $S_1 = 500$ and $S_5 = 3 \times 10^{10}$

(d) 7-order rogue wave with $S_1 = 300$ and $S_6 = 5 \times 10^{10}$

FIGURE 9. (Color online) A *triangular* pattern in a ring for higher order rogue wave with $S_{k-1} \neq 0$ and $S_1 \neq 0$.

(a) (b)

FIGURE 10. (Color online) (a) The *multi-ring* model of 5-order rogue wave solution with $S_2 = 5000$ and $S_4 = 1 \times 10^8$. (b) The local central profile.
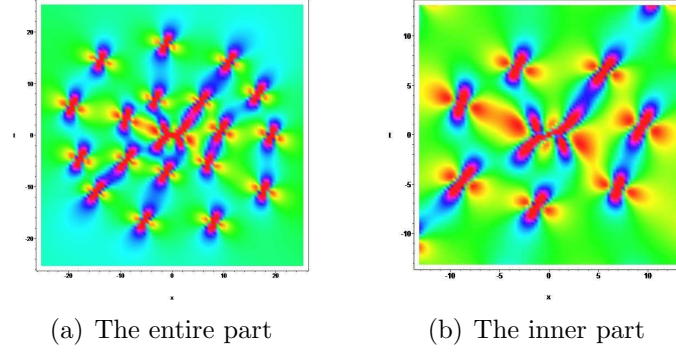
(a) The entire part   (b) The inner part

FIGURE 11. (Color online)The *multi-ring-1* structure of 6-order rogue wave .
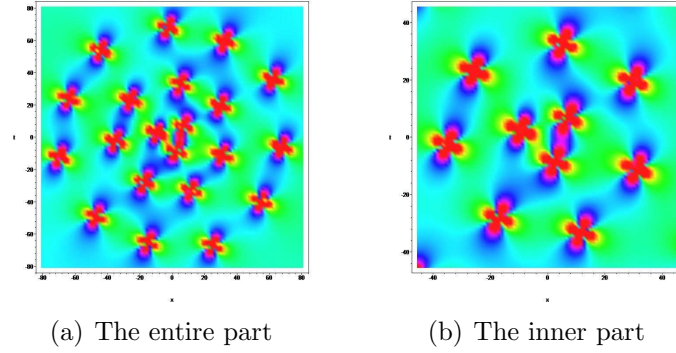

(a) The entire part   (b) The inner part

FIGURE 12. (Color online)The *multi-ring-2* structure of the 6-order rogue wave. Both the outer shell and the middle shell are circular, and the inner shell is triangular(or circular).

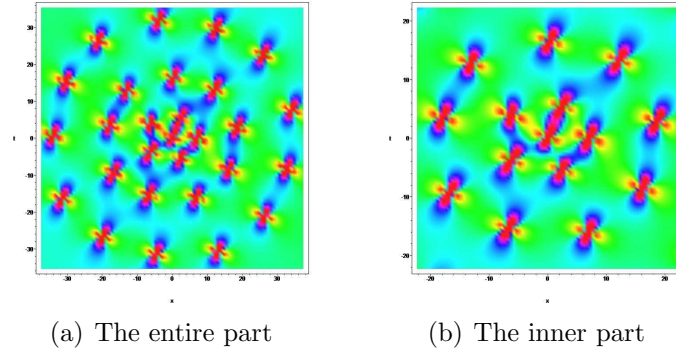(a) The entire part   (b) The inner part

FIGURE 13. (Color online) The *multi-ring-1* structure of 7-order rogue wave with $S_2 = 1000$, $S_4 = 1 \times 10^7$ and $S_6 = 5 \times 10^{11}$. All the three shell are circular.

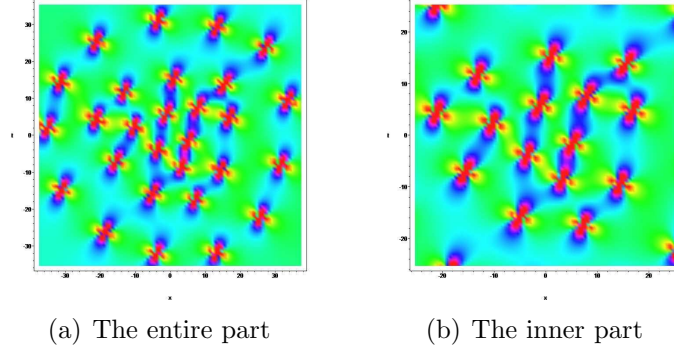(a) The entire part   (b) The inner part

FIGURE 14.   (Color online) The *multi-ring-2* structure of 7-order rogue wave with $S_1 = 100$, $S_4 = 1 \times 10^7$ and $S_6 = 5 \times 10^{11}$. Both the outer shell and the middle shell are circular, and the inner shell is triangular.